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ON THE GRADING NUMBERS OF DIRECT PRODUCTS OF CHAINS

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For a finite lattice L, denote by $l^*(L)$ and $l_*(L)$ respectively the upper length and lower length of L. The grading number $g(L)$ of L is defined as $g(L) = l^*(\mathrm{Sub}(L)) - l_*(\mathrm{Sub}(L))$ where $\mathrm{Sub}(L)$ is the sublattice-lattice of L. We show that if K is a proper homomorphic image of a distributive lattice L, then $l_*(\mathrm{Sub}(K)) < l_*(\mathrm{Sub}(L))$; and derive from this result, formulae for $l_*(\mathrm{Sub}(L))$ and $g(L)$ where L is a product of chains.

1. Introduction

Every lattice is assumed to be finite throughout this note. For a lattice L, we shall denote by $\mathrm{Sub}(L)$ the lattice of all sublattices of L (inclusive of the empty sublattice) under inclusion. We call $\mathrm{Sub}(L)$ the *sublattice-lattice* of L. The *upper length* of $\mathrm{Sub}(L)$, denoted by $l^*(\mathrm{Sub}(L))$, is the length of a longest chain in $\mathrm{Sub}(L)$ and the *lower length* of $\mathrm{Sub}(L)$, denoted by $l_*(\mathrm{Sub}(L))$, is the length of a shortest maximal chain in $\mathrm{Sub}(L)$. The *grading number* of L, denoted by $g(L)$, is defined as

$$g(L) = l^*(\mathrm{Sub}(L)) - l_*(\mathrm{Sub}(L)).$$

The lattice L is said to be *of grade* n if $g(L) = n$.

Let C_t be a t-element chain. Lakser observed in [9] that a lattice is of grade zero if and only if it does not contain $C_2 \times C_3$ as a sublattice. Chen, Koh and Lee characterized in [3] all finite distributive lattices of grade one. Going one step further, Koh and Trance determined completely in [8] all finite distributive lattices of grade two.

In contrast with the upper length $l^*(\mathrm{Sub}(L))$, which has been the subject of much attention lately (see for instance [1, 2, 7, 12]), very little is known about the lower length $l_*(\mathrm{Sub}(L))$ and hence the grading number $g(L)$. It seems very difficult to provide an efficient way to compute $l_*(\mathrm{Sub}(L))$ and $g(\mathrm{L})$ even if L is distributive. In the absence of any general result on $l_*(\mathrm{Sub}(L))$ and $g(L)$, we prove in this note that if K is a proper homomorphic image of a distributive lattice L, then $l_*(\mathrm{Sub}(K)) < l_*(\mathrm{Sub}(L))$; and derive from this the following

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identities:

$$l_*\left(\mathrm{Sub}\left(\prod_{i=1}^{m} C_{n(i)}\right)\right)=\sum_{i=1}^{m} n(i)$$

and

$$g\left(\prod_{i=1}^{m} C_{n(i)}\right)=(1-m)+\sum_{i=1}^{m-1}\left(\sum_{j=i+1}^{m}(n(i)-1)(n(j)-1)\right),$$

where $n(i)\geqslant 2$ for each $i=1,2,\ldots,m$.

2. Basic lemmas

To make this note self-contained, we include in this section several known results which will be used to prove our main results in the next section.

A non-empty sublattice N of L is called a *prime* sublattice of L if $L-N$ is either empty or a sublattice of L. A prime sublattice N of L is called a *minimal* prime sublattice of L if N contains no prime sublattice of L other than itself. The set of all minimal prime sublattices of L will be denoted by $\mathrm{mp}(L)$. We refer to [5] for all the definitions, notation and basic results not given here.

Lemma 1 [4]. *Let M be a subset of a distributive lattice L, not necessarily finite. Then M is a maximal sublattice of L iff $L-M\in\mathrm{mp}(L)$.*

Throughout this note, we shall denote by $\mathscr{L}(FD)$ the class of all finite distributive lattices. The following result is a useful characterization of members in $\mathrm{mp}(L)$ for each L in $\mathscr{L}(FD)$.

Lemma 2 [11, 6]. *Let L be in $\mathscr{L}(FD)$ and $N\subseteq L$ with $N\neq\{0\}$ or $\{1\}$. Then $N\in\mathrm{mp}(L)$ iff for each $e\in N$, there exist a unique f and a unique g in $L-N$ such that $g\prec e\prec f$ in L.*

Let A, B be isomorphic sublattices of a lattice L. We shall write $A\ll B$ iff there exists an isomorphism φ from A onto B such that $a\prec a\varphi$ in L for each a in A. Then we have:

Lemma 3 [11, 6]. *Let L be in $\mathscr{L}(FD)$ and A, a sublattice of L. If $A\subseteq N$ for some N in $\mathrm{mp}(L)$ with $N\neq\{0\}$ or $\{1\}$, then there exist A', A^* in $\mathrm{Sub}(L)$ such that $A'\ll A\ll A^*$.*

For any positive integer n, let B_n denote the Boolean lattice with n atoms. We have:

Lemma 4 [10]. *Let N be in $\mathrm{mp}(B_n)$ where $n\geqslant 2$. Then*

$$B_n-N\cong B_{n-2}\times C_3.$$

The *upper length* of Sub(L) is defined as

$$l^*(\mathrm{Sub}(L)) = \mathrm{Max}\{|\Gamma| - 1 \mid \Gamma \text{ is a chain in } \mathrm{Sub}(L)\}.$$

The following results on the upper length of Sub(L) can be found in [7].

Lemma 5. *Let $m \geqslant 1$ and for $i = 1, 2, \ldots, m$, let L_i be in $\mathscr{L}(FD)$. Then*

$$l^*\left(\mathrm{Sub}\left(\prod_{i=1}^{m} L_i\right)\right) = \sum_{i=1}^{m} l^*(\mathrm{Sub}(L_i)) + \sum_{i=1}^{m-1}\left(\sum_{j=i+1}^{m} l(L_i)l(L_j)\right) + (1-m),$$

where $l(L_i)$ is the length of L_i.

In particular, we have:

Corollary. *Let $m \geqslant 1$ and for $i = 1, 2, \ldots, m$, let $C_{n(i)}$ be an $n(i)$-element chain where $n(i) \geqslant 2$. Then*

$$l^*\left(\mathrm{Sub}\left(\prod_{i=1}^{m} C_{n(i)}\right)\right) = (1-m) + \sum_{i=1}^{m} n(i) + \sum_{i=1}^{m-1}\left(\sum_{j=i+1}^{m} (n(i)-1)(n(j)-1)\right)$$

and

$$l^*(\mathrm{Sub}(B_m)) = \tfrac{1}{2}(m^2 + m + 2).$$

3. Main results

We shall now establish in this section our main results as stated in the introduction. Recall that the *lower length* of Sub(L) is defined as

$$l_*(\mathrm{Sub}(L)) = \mathrm{Min}\{|\Gamma| - 1 \mid \Gamma \text{ is a maximal chain in } \mathrm{Sub}(L)\}.$$

We first compare $l_*(\mathrm{Sub}(L))$ with $l_*(\mathrm{Sub}(K))$ where L is in $\mathscr{L}(FD)$ and K is a homomorphic image of L. To this end, we need the following lemma.

Lemma 6. *Let L, K be in $\mathscr{L}(FD)$ and N be in* mp(L). *Let $f: L \to K$ be a homomorphism. If the restriction $f|_{L-N}$ of f to $L-N$ is injective, then $f|_N$ is also injective.*

Proof. Suppose to the contrary that there exist distinct elements p, q in N such that $f(p) = f(q)$. Without loss of generality, we may assume that $p > q$. Since $|N| \geqslant 2$, there exists by Lemma 3 an isomorphic copy N' of N in L such that $N' \ll N$. Let p', q' be in N' with $p' < p$ and $q' < q$ in L. Note that $p' \wedge q = q'$. Observe that

$$f(p') = f(p' \wedge p) = f(p') \wedge f(p) = f(p') \wedge f(q) = f(p' \wedge q) = f(q'),$$

which however contradicts the assumption that $f|_{L-N}$ is injective. Hence we conclude that $f|_N$ is injective.

We now have:

Theorem 1. *Let L and K be in $\mathscr{L}(FD)$. If K is a homomorphic image of L with $|K|<|L|$, then $l_*(\mathrm{Sub}(K))<l_*(\mathrm{Sub}(L))$.*

Proof. Let $f: L\to K$ be an epimorphism. Since $|K|<|L|$, f is not injective.

Claim 1. *If $A\succ B$ in $\mathrm{Sub}(L)$, then either $f(A)=f(B)$ or $f(A)\succ f(B)$ in $\mathrm{Sub}(K)$.*

Indeed, we always have $f(A)\supseteq f(B)$ in K. Suppose that $f(A)\neq f(B)$ and let $D\in\mathrm{Sub}(K)$ such that $f(A)\supseteq D\supseteq f(B)$. Let $C=f^{-1}(D)\cap A$. Then $C\in\mathrm{Sub}(L)$ and $A\supseteq C\supseteq B$. Since A covers B in $\mathrm{Sub}(L)$, we have either $A=C$ or $B=C$. If $A=C$, then $f^{-1}(D)\supseteq A$ and thus $D=f(f^{-1}(D))\supseteq f(A)$. Hence $D=f(A)$. Assume now that $B=C$. Let $x\in D$. Since $D\subseteq f(A)$, there exists a in A such that $f(a)=x$. Thus, $a\in f^{-1}(D)\cap A=C=B$. Hence $x=f(a)\in f(B)$, which implies that $D\subseteq f(B)$ and so $D=f(B)$, establishing Claim 1.

Now let $\emptyset=L_0\prec L_1\prec\cdots\prec L_k=L$ be a maximal chain in $\mathrm{Sub}(L)$.

Claim 2. *$f(L_h)=f(L_{h+1})$ for some $h=1,2,\ldots,k-1$.*

We note that $|L_1|=1$. Thus f is injective on L_1. Since f is not injective on L, there exists an integer $h=1,2,\ldots,k-1$ such that f is injective on L_i for $i=1,2,\ldots,h$ and f is not injective on L_{h+1}. Let $N=L_{h+1}-L_h$. Then $N\in\mathrm{mp}(L_{h+1})$ by Lemma 1. Since $f|_{L_{h+1}-N}=f|_{L_h}$ is injective, by Lemma 6, $f|_N$ is also injective. As $f|_{L_{h+1}}$ is not injective and $L_{h+1}=L_h\cup N$, there exists p in N such that $f(p)\in f(L_h)$. Observe that $p\in f^{-1}(f(L_h))\cap L_{h+1}$ while $p\notin L_h$. Hence we have

$$L_h\subset f^{-1}(f(L_h))\cap L_{h+1}\subseteq L_{h+1},$$

which implies that $f^{-1}(f(L_h))\cap L_{h+1}=L_{h+1}$, i.e., $L_{h+1}\subseteq f^{-1}(f(L_h))$, as L_h is covered by L_{h+1} in $\mathrm{Sub}(L)$. Thus $f(L_{h+1})\subseteq f(L_h)$ and so $f(L_h)=f(L_{h+1})$.

Combining Claims 1 and 2, we conclude that $l_*(\mathrm{Sub}(K))<l_*(\mathrm{Sub}(L))$, as required.

From Lemma 3, we see that if N is in $\mathrm{mp}(L)$ with $N\neq\{0\}$ or $\{1\}$, then $[a,b]$ contains a sublattice isomorphic to $N\times C_3$ where $a=\bigwedge N'$ and $b=\bigvee N^*$. If it happens that $[a,b]\cong N\times C_3$, then we call N a *cut* of L.

The following result is an immediate consequence of Lemma 1 and Theorem 1.

Corollary. *Let L be in $\mathscr{L}(FD)$ and N, a cut of L. Then $L-N$ is a homomorphic image of L and $l_*(\mathrm{Sub}(L))=l_*(\mathrm{Sub}(L-N))+1$.*

We are now in a position to establish the following theorem.

Theorem 2. *Let $m \geqslant 1$ and for $i = 1, 2, \ldots, m$, let $C_{n(i)}$ be an $n(i)$-element chain where $n(i) \geqslant 2$. Then*

(i) $$l_*\left(\mathrm{Sub}\left(\prod_{i=1}^{m} C_{n(i)}\right)\right) = \sum_{i=1}^{m} n(i)$$

and

(ii) $$g\left(\prod_{i=1}^{m} C_{n(i)}\right) = (1-m) + \sum_{i=1}^{m-1}\left(\sum_{j=i+1}^{m} (n(i)-1)(n(j)-1)\right).$$

Proof. We shall prove the result (i) by induction on $r(=\sum_{i=1}^{m} n(i) \geqslant 2)$. If $r = 2$, then $m = 1$ and it is clear that $l_*(\mathrm{Sub}(C_2)) = 2$. Assume that (i) is true when $r = k-1$. Consider $r = \sum_{i=1}^{m} n(i) = k$ and let $L = \prod_{i=1}^{m} C_{n(i)}$. If L is a Boolean lattice, we choose a cut N of L as in Lemma 4. By the corollary to Theorem 1, we have $l_*(\mathrm{Sub}(L)) = l_*(\mathrm{Sub}(L-N)) + 1$. If L is not a Boolean lattice, then there exists an element a in $C_{n(1)}$ (say) such that $\bigwedge C_{n(1)} < a < \bigvee C_{n(1)}$. Let $N = \{a\} \times \prod_{i=2}^{m} C_{n(i)}$. Note that N, considered as a sublattice of L, is a cut of L. Thus by the corollary to Theorem 1 again, we have $l_*(\mathrm{Sub}(L)) = l_*(\mathrm{Sub}(L-N)) + 1$. In both cases, $L - N$ is a product of chains whose sum of cardinalities is $k-1$. Thus by induction hypothesis, $l_*(\mathrm{Sub}(L)) = (k-1) + 1 = k$.

The identity (ii) now follows immediately from the corollary to Lemma 5 and the identity (i).

To end this note we would like to raise the following two problems.

Problem 1. Let L and K be in $\mathscr{L}(FD)$. Is it always true that

$$l_*(\mathrm{Sub}(L \times K)) = l_*(\mathrm{Sub}(L)) + l_*(\mathrm{Sub}(K))?$$

The equality holds if both L and K are products of chains by Theorem 2(i), and up till now we are still unable to find a counterexample.

Problem 2. The lattice $\mathrm{Sub}(L)$ is said to be *uniform* if for each integer k with $l_*(\mathrm{Sub}(L)) \leqslant k \leqslant l^*(\mathrm{Sub}(L))$, there is a maximal chain Γ of length k in $\mathrm{Sub}(L)$. Is the lattice $\mathrm{Sub}(L)$ always uniform for each L in $\mathscr{L}(FD)$? No answer is known even if L is a product of chains.

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